Paper B2: Radiation and Matter - Basic Laser Physics

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Lecture 1

The Interaction of Radiation and Matter

1.1 Introduction

In these four lectures we will look at the basic physics of laser systems. I would be very grateful if you were to bring any errors or unclear passages to my attention (email: simon.hooker@physics.ox.ac.uk).

1.2 Radiation

Last year in Statistical Mechanics you will have investigated the properties of radiation, and in particular blackbody radiation. Here we review the main findings in preparation for a discussion of the interaction of radiation and matter.

1.2.1 Radiation modes

By examining the possible solutions to Maxwell’s Equations in a cube of side $L$, it is found that the number of allowed modes with angular frequency in the range $\omega$ to $\omega + d\omega$ is proportional to the volume of the cube. In fact, provided a cavity is large enough, the number of such modes per unit volume is independent of the size or shape of the cavity, and, indeed, the material of the cavity walls. Free space may be thought of as the limit $L \rightarrow \infty$ such that the allowed modes are distributed essentially continuously in frequency with a mode density given by

$$g(\omega)d\omega = \frac{\omega^2}{\pi^2 c^3} d\omega.$$  \hspace{1cm} (1.1)

\footnote{This is our first example of a spectral quantity, that is a quantity per unit frequency interval, in this case the density of radiation modes per unit frequency interval. Spectral quantities play an important role in the theoretical description of the interaction of radiation and matter.}
1.2.2 Planck’s Law

Classically the energy of a radiation mode is given by the following result from Electromagnetism (assuming a linear medium):

\[
W_{\text{mode}} = \int_{\text{vol}} \left( \frac{1}{2} \mathbf{E} \cdot \mathbf{D} + \frac{1}{2} \mathbf{B} \cdot \mathbf{H} \right) d\tau,
\]

where the bar indicates an average over time. Clearly this can take any positive value since the amplitudes of the electric and magnetic fields are not constrained.

These purely classical considerations, however, do not provide a complete description of the radiation field and we should instead move to a quantum mechanical picture in which the electromagnetic fields are also quantized. Such an approach is beyond the scope of the course, and we may instead adopt the hypothesis of Planck. We assume that since the electromagnetic fields of each mode oscillates harmonically, the energy stored in the fields are quantized as they would be for any other quantum mechanical oscillator. In other words, we must have

\[
W_{\text{mode}} = (n + 1/2)\hbar \omega, \quad n = 1, 2, 3, \ldots
\]

Within this picture the radiation field is said:

1. “to be in the nth excited state,” or,
2. “to contain n quanta of energy,” or,
3. “to contain n photons”.

The distribution of photons over the modes is given by the Boltzmann distribution, i.e. for radiation with a temperature \(T\), the probability of the mode containing \(n\) photons is given by,

\[
P_n = \frac{\exp\left(-\frac{(n + 1/2)\hbar \omega}{k_B T}\right)}{\sum_{n=0}^{\infty} \exp\left(-\frac{(n + 1/2)\hbar \omega}{k_B T}\right)}
\]

The mean number of photons in the mode is then found straightforwardly to be,

\[
\bar{n}_\omega = \frac{1}{\exp(h \omega / k_B T) - 1}
\]

Now, in general the energy density of the radiation field for angular frequencies in the range \(\omega, \omega + d\omega\) is given by:

\[2\text{For further information consult The Quantum Theory of Light by R. Loudon, Oxford University Press (ISBN 0198501765).}\]

\[3\text{In fact Planck assumed that the walls of the cavity were composed of oscillators that could only absorb or emit discrete quanta of energy. He did not, as such, consider the electromagnetic radiation to be quantized. Note also that Planck took the energy of the oscillators to be of the form } n\hbar \omega, \text{ and hence did not include the zero-point energy.}\]

\[4\text{Another example of a spectral quantity. Here } \rho(\omega)d\omega \text{ is the energy density of the radiation field per unit frequency interval. We should be aware that often we wish to deal with non-spectral quantities, e.g. the total energy density of the radiation field } U = \int \rho(\omega)d\omega. \text{ Where confusion might arise between spectral and non-spectral quantities we will write spectral quantities in the form } a(\omega)d\omega.\]
1.2. RADIATION

Figure 1.1: (a) a collimated beam of radiation; (b) isotropic radiation.

\[ \rho(\omega) d\omega = \bar{n}_\omega \times g(\omega) d\omega \times \hbar \omega. \]  
(1.6)

Equation (1.6) is always true. For the particular case radiation in equilibrium with matter at temperature \( T \), known as blackbody radiation, combining eqns (1.6) and (1.5) gives,

\[ \rho_B(\omega) d\omega = \frac{\hbar \omega^3}{\pi^2 c^3} \frac{d\omega}{\exp(\hbar \omega/k_B T) - 1}. \]  
(1.7)

This is Planck’s Law\(^5\).

1.2.3 Comparison of number of photons per mode for different sources

It is useful at this point to compare the number of photons per mode for a thermal source and a laser. In order to do this we must relate experimentally measured quantities, such as intensity, to the quantities appearing in eqn. (1.6). This relation depends on the spatial distribution of the radiation, the two most important cases being a collimated beam or isotropic radiation.

Collimated beam

Consider the collimated beam shown schematically in Figure 1.1(a) of \( n(\omega) \delta\omega \) photons per unit volume in the frequency interval \( \omega, \omega + \delta\omega \). In time \( \delta t \), \( n(\omega) \delta\omega \cdot c\delta t \) photons in this frequency interval cross unit area. Each of these photons carries an energy \( \hbar \omega \), and hence the spectral intensity (units W m\(^{-2}\) per unit frequency interval) is given by,

\[ I(\omega) \delta\omega = n(\omega) \hbar \omega c \delta\omega, \]

or

\[ I(\omega) = \rho(\omega) c \text{ collimated beam} \]  
(1.8)

\(^5\)The subscript B in eqn. (1.7) reminds us that this expression is only valid for blackbody radiation.
Isotropic radiation

Now consider a small area $A$ within an isotropic radiation field (Fig. 1.1(b)) of $n(\omega)$ photons per unit volume per unit frequency interval. Considering the photons to behave as an ideal gas, Kinetic Theory tells that the number of photons in the frequency interval $\omega, \omega + \delta\omega$ striking one side per unit area per second is $\frac{1}{4} n(\omega) \delta\omega c$. Hence, in this case, the spectral intensity of the radiation is given by,

$$I(\omega) = \frac{1}{4} n(\omega) \hbar \omega c = \frac{1}{4} \rho(\omega) c$$

(1.9)

Some examples

We are now in a position to calculate the number of photons per mode for two examples. To do this we relate the total intensity $I_T$ of the source (units W m$^{-2}$) to the spectral intensity through $I_T = I(\omega) \Delta\omega$, where $\Delta\omega$ is the bandwidth of the source. From the spectral intensity we can find $n(\omega)$ the spectral number density of photons from eqn. (1.8) or (1.9). The number of photons per mode is then given by $\bar{n}_\omega = n(\omega)/g(\omega)$, where $g(\omega)$ is given by eqn. (1.1).

We consider two examples:

**Single-isotope mercury lamp** Use of a single isotope of mercury keeps the line broadening to the minimum possible set by the Doppler effect. The narrowest bandwidth that might be achievable is approximately 0.2 nm for the resonance line at 253.7 nm. By focusing with a good lens we can achieve an intensity of order 1 W cm$^{-2}$.

**Frequency-stabilized argon-ion laser** A small argon-ion laser can deliver 5 W of power in a beam of 2 mm diameter. With proper frequency stabilization the linewidth of the 488 nm line can be reduced to approximately 0.002 nm.

Table 1.1 summarizes this calculation. We see that for the mercury lamp the mean number of photons per mode is significantly less than unity. In contrast, for the laser this quantity is much greater than one. In fact $\bar{n}_\omega$ is always found to be $\gg 1$ for lasers and $\ll 1$ for non-laser source. As we will see later, this difference reflects the relative importance of stimulated and spontaneous emission in the sources.

1.3 The Einstein description

In 1917 Einstein introduced a phenomenological model of the interaction of radiation and matter. Quantum mechanical calculations of this interaction are in complete agreement with the Einstein picture, provided certain conditions are met concerning the strength of the radiation field and the linewidth of the transition. In practice the Einstein approach is found to be valid for the conditions found inside virtually all lasers, leading to the use of so-called rate equations to describe the population densities in the energy levels of the laser.
Table 1.1: Spectral parameters of two sources.

<table>
<thead>
<tr>
<th></th>
<th>Hg lamp</th>
<th>Ar$^+$ laser</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wavelength (nm)</td>
<td>$\lambda$</td>
<td>253.7</td>
</tr>
<tr>
<td>Bandwidth (nm)</td>
<td>$\Delta\lambda$</td>
<td>0.2</td>
</tr>
<tr>
<td>Frequency bandwidth (rad s$^{-1}$)</td>
<td>$\Delta\omega$</td>
<td>$6 \times 10^{12}$</td>
</tr>
<tr>
<td>Intensity (W m$^{-2}$)</td>
<td>$I_T$</td>
<td>$10^4$</td>
</tr>
<tr>
<td>Spectral intensity (W m$^{-2}$ per freq. int.)</td>
<td>$I(\omega)$</td>
<td>$1.7 \times 10^{-9}$</td>
</tr>
<tr>
<td>Spectral photon density (photons m$^{-3}$ per freq. int.)</td>
<td>$n(\omega)$</td>
<td>7.3</td>
</tr>
<tr>
<td>Spectral mode density (modes m$^{-3}$ per freq. int.)</td>
<td>$g(\omega)$</td>
<td>$2.1 \times 10^5$</td>
</tr>
<tr>
<td>Photons per mode</td>
<td>$n_{\omega}$</td>
<td>$3 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

Einstein considered two levels of an atom, an upper level of energy $E_2$, and a lower level of energy $E_1$. He identified three processes by which radiation could interact with atoms in these levels. The first is spontaneous emission, in which an atom in the upper level decays to the lower level by the emission of a photon with energy $\hbar \omega_{21} = E_2 - E_1$. The spontaneously emitted photon can be emitted in any direction.

The second and third processes are absorption, in which an atom in the lower level is excited to the upper level by the absorption of a photon of energy $\hbar \omega_{21}$; and stimulated emission, in which an incident photon of energy $\hbar \omega_{21}$ stimulates an atom in the upper level to decay to the lower level by the emission of a second photon of energy $\hbar \omega_{21}$. The stimulated photon is emitted into the same mode as the incident photon, and hence has the same frequency, direction, and polarization as the incident photon. This third process, stimulated emission, is the key to the operation of the laser.

It seems reasonable that the rate of spontaneous emission should be independent of the conditions of the radiation field in which the atom finds itself. Furthermore, it is clear that the rates at which absorption and stimulated emission occur must depend in some way on the density of photons of energy $\hbar \omega_{21}$, or, equivalently, on the energy density of the radiation field at $\omega_{21}$. In three postulates Einstein went further and stated that the rates of absorption and stimulated emission are linearly dependent on the energy density at $\omega_{21}$. The Einstein postulates may be stated as follows:

1. The rate per unit volume at which atoms in the upper level 2 decay spontaneously to the lower level 1 is equal to $N_2 A_{21}$, where $N_2$ is the number of atoms per unit volume in level 2, and $A_{21}$ is a constant characteristic of the transition;

2. The rate per unit volume at which atoms in the lower level are excited to the upper level by the absorption of photons of energy $\hbar \omega_{21}$ is equal to $N_1 B_{12} \rho(\omega_{21})$, where $N_1$ is the number of atoms per unit volume in level
Figure 1.2: Illustration of the interaction of radiation with two levels of an atom by spontaneous emission, absorption, and stimulated emission. For each process the transition rates are given in terms of the Einstein coefficients and the number densities $N_2$ and $N_1$ of atoms in the upper and lower levels. The photons have energy $\hbar \omega_{21}$ and the figure should be read from left to right. For example, in the case of stimulated emission an incident photon of energy $\hbar \omega_{21}$ stimulates an excited atom to make a transition to the lower level, emitting a second photon in the process.

1. $\rho(\omega_{21})$ is the energy density of radiation of angular frequency $\omega_{21}$ and $B_{12}$ is a constant characteristic of the transition;

3. The rate per unit volume at which atoms in the upper level decay to the lower level by the stimulated emission of photons of energy $\hbar \omega_{21}$ is equal to $N_2B_{21}\rho(\omega_{21})$, where $B_{21}$ is a constant characteristic of the transition.

The coefficients $A_{21}$, $B_{12}$, and $B_{21}$ are known as the Einstein A and B coefficients. The three fundamental interactions between atoms and radiation are shown schematically in Fig. 1.2.

Note: The energy density $\rho(\omega_{21})$ appearing in the definition of the Einstein coefficients is a spectral quantity, but the population densities are total population densities (having units of atoms per unit volume). Hence the units of $A_{21}$ are simply s$^{-1}$, whilst those of $B_{12}$ and $B_{21}$ are m$^3$J$^{-1}$rad$^{-2}$.

There is an important subtlety here. We have assumed that the levels of the atom have perfectly defined energies, and as such only interact with radiation at exactly $\omega_{21}$. In practice, however, the atomic levels will always be broadened to some extent. We will see later how to adapt the Einstein approach to deal with broadened levels.

### 1.3.1 Relations between the Einstein coefficients

For a given transition the three Einstein coefficients are postulated to be constant, that is they depend upon the atom, and the energy levels of the atom that are involved, but they are independent of the radiation field. We can use this fact to find relations between the Einstein coefficients by considering a special case in which the relative populations of the upper and lower levels are known.

We consider an ensemble of stationary atoms immersed in a bath of black-body radiation of temperature $T$ and energy density $\rho(\omega)$. The reason for so
1.3. THE EINSTEIN DESCRIPTION

Figure 1.3: Illustrating the dynamic thermal equilibrium between levels in an atom. Atoms in any level $k$ may undergo radiative transitions to a large number of higher and lower levels. However, the Principle of Detailed Balance states that the population of any pair of levels $j, k$ must be in dynamic equilibrium.

Doing is that we know: (i) the distribution of population over the energy levels (from Boltzmann); (ii) we know the energy density of the radiation is given by Planck’s Law; (iii) we can then compare the results expected from (i) and (ii) with that predicted by the Einstein treatment.

In general the atoms will have many energy levels $E_i, E_j, E_k...$ with degeneracies $g_i, g_j, g_k...$. Now, atoms in any level $j$ will, in principle, undergo transitions to all higher levels $k$ by absorbing radiation of angular frequency $\omega_{kj}$, where $\hbar \omega_{kj} = E_k - E_j$, as well as making transitions to all lower levels $i$ by spontaneous and stimulated emission of radiation of angular frequency $\omega_{ji}$, where $\hbar \omega_{ji} = E_j - E_i$. The atoms will be in a dynamic equilibrium such that, whilst an individual atom will continually make transitions between its various energy levels, the total number of atoms in a given level remains, on average, constant.

Clearly atoms in any level can make transitions to a large, even infinite, number of other levels. The Principle of Detailed Balance states, however, that in thermal equilibrium the transitions between any pair of levels are also in dynamic equilibrium, as illustrated schematically in Fig. 1.3.

The reasoning behind the Principle of Detailed Balance is most easily seen by considering the radiation field, rather than the populations of the atomic levels. Since the system is in thermodynamic equilibrium, it must be that the rate at which the radiation field loses photons of angular frequency $\omega$ by absorption must be balanced by the rate at which it gains photons of this frequency by spontaneous and stimulated emission. Now an arbitrary pair of atomic levels 1, 2 will only absorb or emit photons of a particular frequency, $\omega_{21}$, and this frequency is unique to that pair of levels. It must be, then, that transitions between levels 1 and 2, and that pair alone, maintain the density of photons of frequency $\omega_{21}$.

In terms of the Einstein coefficients, equating the rate of transitions from
2 → 1 with that for transitions from 1 → 2 yields:

\[ N_2 B_{21} \rho_B(\omega_{21}) + N_2 A_{21} = N_1 B_{12} \rho_B(\omega_{21}) \]  
\( \text{(1.10)} \)

from which we find,

\[ \rho_B(\omega_{21}) = \frac{A_{21}/B_{21}}{N_2 B_{21} - 1} \]  
\( \text{(1.11)} \)

However, since the system is in thermal equilibrium we know that the population ratio is given by the Boltzmann equation,

\[ \frac{N_2}{N_1} = \frac{g_2}{g_1} \exp \left( -\frac{\hbar \omega_{21}}{k_B T} \right), \]  
\( \text{(1.12)} \)

and hence eqn. (1.11) can be re-written,

\[ \rho_B(\omega_{21}) = \frac{A_{21}/B_{21}}{\frac{g_2}{g_1} \exp \left( \frac{\hbar \omega_{21}}{k_B T} \right) - 1} \]  
\( \text{(1.13)} \)

This last result must be consistent with Planck’s Law (eqn. (1.7)) for all temperatures \( T \). We therefore deduce that,

\[ g_1 B_{12} = g_2 B_{21} \]  
\( \text{(1.14)} \)

\[ A_{21} = \frac{\hbar \omega_{21}^3}{\pi^2 c^3} B_{21}. \]  
\( \text{(1.15)} \)

These are the well known Einstein relations for the A and B coefficients.

### 1.4 Conditions for gain

As we will see more formally later, for light amplification by the stimulated emission of radiation we require the rate of stimulated emission to be greater than the rate of absorption:

\[ N_2 B_{21} \rho(\omega_{21}) > N_1 B_{12} \rho(\omega_{21}) \]

\( \Rightarrow \frac{N_2}{g_2} > \frac{N_1}{g_1} \quad \text{Condition for optical gain} \)  
\( \text{(1.16)} \)

In other words, the population per state must be greater in the upper level than in the lower level, a situation called a population inversion.

We should realize that a population inversion is unusual. If the level populations, for example, were in thermal equilibrium at temperature \( T \) the populations of the levels would be described by a Boltzmann distribution:
1.4. CONDITIONS FOR GAIN

Figure 1.4: Comparison of the populations per state for (a) an atom in thermal equilibrium; (b) an atom with a population inversion between levels 2 and 1.

\[
\frac{N_2}{N_1} = \frac{g_2}{g_1} \exp \left( - \frac{E_2 - E_1}{k_B T} \right), \tag{1.17}
\]

or,

\[
\left( \frac{N_2}{g_2} \right) \left( \frac{N_1}{g_1} \right) = \exp \left( - \frac{(E_2 - E_1)}{k_B T} \right). \tag{1.18}
\]

from which it is clear that for a system in thermal equilibrium the population per state of an upper level is always lower than that for any lower-lying level. Figure 1.4 illustrates schematically the distribution over levels for (a) thermal equilibrium and (b) a population inversion between levels 2 and 1.

1.4.1 Conditions for steady-state inversion

Having demonstrated that a population inversion cannot exist under conditions of thermal equilibrium, it is useful to explore the conditions under which a population inversion can be produced. Figure 1.5 shows schematically the kinetic processes which affect the populations of the upper and lower levels of a laser transition. We suppose that atoms in the upper level are produced, or ‘pumped’ at a rate \( R_2 \) (units of atoms m\(^{-3}\) s\(^{-1}\)), and that the lifetime of the upper level is \( \tau_2 \). Note that the pump rate includes all processes that excite the upper level such as direct optical pumping, electron collisional excitation, and radiative and non-radiative cascade from higher-lying levels. The lifetime \( \tau_2 \) is the lifetime against all types of decay (radiative, collisional de-excitation, etc.) and includes radiative decay to level 1. It is known as the fluorescence lifetime, since it is the lifetime with which the strength of the fluorescence on any radiative transition from level 2 would be observed to decay. We define the pump rate and lifetime of the lower laser level in a similar way, but do not include spontaneous emission on the laser transition itself in \( R_1 \). That contribution to the population of level 1 will be handled explicitly, for reasons that will be clear below.

The evolution of the population densities in the two levels may then be written as,
LEC TURE 1. THE INTERACTION OF RADIAT ION AND MATTER

\[ \frac{dN_2}{dt} = R_2 - \frac{N_2}{\tau_2} \]  
\[ \frac{dN_1}{dt} = R_1 + N_2 A_{21} - \frac{N_1}{\tau_1}. \]

Note that the symmetry of the two equations is broken by the spontaneous emission term, \( N_2 A_{21} \). It is straightforward to solve the above for steady-state conditions by setting \( dN_2/dt = dN_1/dt = 0 \). We find,

\[ N_2 = R_2 \tau_2 \]  
\[ N_1 = R_1 \tau_1 + R_2 \tau_2 A_{21} \tau_1. \]

For optical gain we require \( N_2/g_2 > N_1/g_1 \) which yields the following condition for a steady-state population inversion:

\[ \frac{R_2 \tau_2 g_1}{R_1 \tau_1 g_2} \left[ 1 - \frac{g_2}{g_1} A_{21} \tau_1 \right] > 1. \textbf{Steady-state inversion} \]

We conclude, therefore, that for a steady-state population inversion to be achieved at least one of the following must be true:

**Selective pumping** \( R_2 > R_1 \): in which the upper laser level is pumped more rapidly than the lower laser level;

**Favourable lifetime ratio** \( \tau_2 > \tau_1 \): in which the lower laser level decays more rapidly than the upper level, which keeps the population of the lower level small;

**Favourable degeneracy ratio** \( g_1 > g_2 \): which ensures that the population per state of the lower laser level is small.
1.4. CONDITIONS FOR GAIN

1.4.2 Necessary, but not sufficient condition

The factor in square brackets in eqn. (1.23) is interesting in that it can be negative as well as positive and is independent of the pumping rates, depending only on the parameters of the laser transition. In other words, for some systems no matter how selective the pumping is it is not possible to achieve a steady-state population inversion. The reason for this somewhat surprising result is that increasing the population of the upper laser level by pumping harder also increases the rate at which the lower level is populated by spontaneous emission on the laser transition itself.

This final factor in eqn. (1.23) therefore yields a necessary, but not sufficient condition for achieving a steady-state population inversion. The condition that this factor is positive can be re-written as,

\[
A_{21} < \frac{g_1}{g_2 \tau_1} \quad \text{necessary, but not sufficient condition} \tag{1.24}
\]

Hence the rate of spontaneous decay from the upper to the lower laser level must be smaller than the total rate of decay from the lower laser level (multiplied by a factor of \(g_1/g_2\)). In other words, the lower level has to empty sufficiently quickly for population not to build up by spontaneous emission on the laser transition.

It should be stressed that satisfying eqn. (1.24) does not ensure that a steady-state population inversion will be achieved on a given transition. For example the pumping may not preferentially populate the upper laser level, or the lifetime ratio may be very unfavourable. However, unless eqn. (1.24) is satisfied, no pumping technique, no matter how selective, will be able to create a steady-state population inversion on the transition. Of course, a transient inversion can always be created with suitably selective pumping.
Lecture 2

The Optical Gain Cross-Section

2.1 Introduction

We saw in the last lecture how to treat the interaction of radiation and matter in terms of the Einstein coefficients. However, in that approach we assumed that the atomic levels were infinitely sharp, and consequently the radiation emitted in transitions between them was purely monochromatic. In reality, of course, that is not the case and for any given transition a number of mechanisms can lead to broadening of the radiation emitted. In this lecture we discuss how to modify the Einstein treatment when spectral broadening is present, and introduce the optical gain coefficient.

2.2 Homogeneous line broadening

In general line broadening mechanisms can be divided into two classes: homogeneous broadening and inhomogeneous broadening. A homogeneous broadening mechanism is so called because it affects all atoms in the sample equally, and consequently all atoms will interact with a beam of radiation of frequency $\omega$ with the same strength. In contrast, an inhomogeneous broadening mechanism causes different atoms to interact with radiation of frequency $\omega$ differently. A good example of an inhomogeneous broadening mechanism is Doppler broadening. Here the transition frequencies of the atoms in a gas are shifted by the Doppler effect according to their relative velocity with respect to the observer (or a beam of radiation). The spectrum emitted by such a sample will be broadened according to the distribution of relative velocities. This broadening is inhomogeneous since only some fraction of atoms (those with a particular velocity towards the observer) emit radiation of frequency $\omega$. Similarly, a narrow-band beam of radiation of frequency $\omega$ will only interact with those atoms which have a velocity towards the beam which is just right to shift their transition frequency into resonance with the beam. In other words the beam will only interact with some fraction of the atoms in the sample.
2.2.1 The classical electron oscillator model of the atom

Useful insight into the mechanisms which lead to homogeneous broadening is obtained from the classical electron oscillator model of the atom familiar from your studies of Electromagnetism.

In this picture the active electron of an atom is imagined to be bound to the atomic nucleus by a harmonic restoring force of the form $-m_e \omega_0^2 r$, where $r$ is the displacement of the electron from its equilibrium position. The motion of the electron is assumed to be damped by a force $-m_e \gamma \dot{r}$, and the nucleus of the atom is considered to be fixed in space.

The great advantage of the classical electron oscillator model is that it is able to reproduce the results of the considerably more complex quantum calculation. Within the classical electron oscillator model the parameter $\omega_0$ is the resonant frequency of the oscillating electron. In terms of the quantum mechanical picture of the interaction of an atom with radiation, $\omega_0$ is the frequency of the transition under consideration.

In the absence of any applied force the equation of motion of the electron may be written as,

$$m_e \frac{d^2 r}{dt^2} = -m_e \omega_0^2 r - m_e \gamma \frac{dr}{dt}$$  \hspace{1cm} (2.1)

In the limit that the damping is small, this has solutions of the form,

$$r(t) = r_0 \exp \left(-\frac{\gamma}{2} t\right) \cos(\omega_0 t).$$  \hspace{1cm} (2.2)

Now, classically the energy of the oscillator is given by,

$$W(t) = \frac{1}{2} m_e \omega_0^2 r^2 + \frac{1}{2} m_e \left(\frac{dr}{dt}\right)^2$$

$$= \frac{1}{2} m_e \omega_0^2 r_0^2 \exp(-\gamma t).$$  \hspace{1cm} (2.3)

We see that the classical oscillator loses energy exponentially with a time constant $1/\gamma$.

Relation to the Einstein A-coefficient

We can relate the classical damping rate to the Einstein A-coefficient by considering an ensemble of atoms initially all excited to an upper level $k$. As illustrated in Fig. 2.1, the atoms can decay spontaneously to a range of lower levels $l, m, n \ldots$. The rate equation for the population of atoms in level $k$ is simply,

$$\frac{dN_k}{dt} = -N_k A_{kl} - N_k A_{km} - N_k A_{kn} - \ldots$$

$$= -N_k \sum_{E_j < E_k} A_{ki}. \hspace{1cm} (2.4)$$
Hence the population of excited atoms decays exponentially according to,

\[ N_k(t) = N_k(0) \exp\left( -\frac{t}{\tau_{\text{rad}}^k} \right), \]

where the **radiative lifetime** is given by,

\[ \frac{1}{\tau_{\text{rad}}^k} = \sum_{E_i < E_k} A_{ki}. \]

Clearly the energy stored in the excited atoms will decay at the same rate, and hence relating this last result to eqn. (2.3) we find,

\[ \gamma = \frac{1}{\tau_{\text{rad}}^k} = \sum_{E_i < E_k} A_{ki}. \]

2.2.2 Lifetime broadening

Within the classical model, an excited atom has a dipole moment \( p(t) = -e\mathbf{r}(t) \) that undergoes a damped oscillation according to eqn. (2.2). At large distances from the atom the amplitude of the electric field of the radiated electromagnetic wave is proportional to \( \dot{p}(t) \), so that we may write the amplitude of the electric field at some distant point as,

\[ \mathbf{E}(t) = \mathbf{E}(0) \exp(-\frac{\gamma}{2}t) \sin(\omega_0 t) \]

The frequency distribution of the oscillating electric field is given by the Fourier Transform of \( \mathbf{E}(t) \), which we will denote by \( \mathbf{E}_{\omega} \). You should be able to see immediately that the spectrum consists of a range of frequencies owing to the fact that the electric field is not a pure harmonic wave. Now, the signal from
any detection system able to measure the emitted spectrum will be proportional to $|E_\omega|^2$. By calculating the Fourier Transform we find,

$$|E_\omega|^2 \propto g_L(\omega - \omega_0) = \frac{1}{\pi} \frac{\gamma/2}{(\omega - \omega_0)^2 + (\gamma/2)^2}.$$  \hspace{1cm} (2.9)

The function $g_L(\omega - \omega_0)$ is known as the lineshape function and is normalized\(^1\) such that $\int_0^\infty g_L(\omega - \omega_0) d\omega = 1$. The full-width at half-maximum (FWHM) of the lineshape is found straightforwardly from eqn. (2.9) to be,

$$\Delta \omega_{\text{N}} = \gamma \hspace{1cm} \text{Natural linewidth}$$ \hspace{1cm} (2.10)

We see that damping of the electron motion leads to the atom emitting a finite range of frequencies, described by a Lorentzian lineshape function as illustrated schematically in Fig. 2.2. The damping is also associated with the excited atom having a finite lifetime $\tau_{\text{rad}}$ and consequently this broadening of the transition is known as lifetime broadening. It must be the case that for emission from an excited level to be observable the level must have a finite radiative lifetime, and will therefore emit radiation with a finite spectral width. For this reason lifetime broadening is often known as natural broadening since it is intrinsic to any radiative transition.

**2.2.3 Pressure broadening**

Radiation emitted by a sample of atoms in the gaseous phase will be homogeneously broadened as a result of collisions with other atoms, electrons, ions etc. This is known as pressure broadening.

We may extend the classical model to the case of pressure broadening in an intuitively obvious way. We suppose that a radiating atom emits a damped wave of the form of eqn. (2.8) until such time it experiences a collision at a time

---

\(^1\)Actually the function $g_L(x)$ is normalized such that $\int_{-\infty}^{\infty} g_L(x) dx = 1$. In eqn (2.9) the argument of $g_L(x)$ extends only to $-\omega_0$. However, since the linewidth is always very small compared to $\omega_0$, extending the lower limit to $-\infty$ will have negligible effect on the integral.
2.3. PHONON BROADENING

Figure 2.3: The classical model of collision broadening. A damped oscillating dipole is subject to a collision, leading to the electric field shown in (a). When averaged over the distribution of collision times, this gives rise to a frequency spectrum with a Lorentzian profile (b).

$\tau_i$ whereupon the atom stops radiating. As before, the frequency distribution of the radiated wave is given by the Fourier transform of the radiated electric field. Now, however, the spectrum will be modified and, in particular, will depend on how long the atom radiates before it makes a collision. From our knowledge of Fourier Transforms we can guess that if $\tau_i$ is short the spectrum will be wide in frequency, whereas if the atom radiates for a long time before suffering a collision the frequency width will essentially be the natural width.

Now, the spectrum observed from a macroscopic sample of atoms will be given by averaging the spectra over the distribution of times between collisions. From Kinetic Theory we know that these times are distributed according to

$$P(\tau_i)d\tau_i = \exp\left(-\frac{\tau_i}{\tau_c}\right)\frac{d\tau_i}{\tau_c}, \quad (2.11)$$

where $\tau_c$ is the mean time between collisions.

Working through this analysis we find that the lineshape is once more Lorentzian, but now the width is modified to,

$$\Delta \omega_p = \gamma + \frac{2}{\tau_c} \quad \text{Pressure-broadened linewidth} \quad (2.12)$$

Note the factor of 2 in the second term.

Clearly the linewidth in this case increases as the mean time between collisions decreases. Typically $\tau_c$ is inversely proportional to the pressure of the gas, in which case the increase in linewidth is proportional to pressure. For obvious reasons, this type of broadening is known as pressure broadening.

The process is illustrated schematically in Fig. 2.3

2.3 Phonon broadening

In many solid-state lasers the active species are ions doped into a crystalline host. The energy level structure of ions in a crystalline environment is generally very different than that of an isolated ion owing to interactions with the
surrounding ions of the lattice. These interactions can be described in terms of a crystal electric field which, for a perfect crystal at zero temperature, will have a symmetry reflecting that of the crystal lattice.

The crystal field splits and shifts the energy levels from their positions in the isolated ion to give a rich energy level structure. Since the positions of the energy levels depends on the exact locations of the neighbouring ions, thermal motion of the crystal lattice causes the energy levels of the dopant ion to fluctuate about their zero-temperature positions. Since the time-scale of thermal oscillations of the lattice is very fast, the energy levels appear to be smeared out, or broadened. Broadening of a large number of close-lying levels split by the crystal field can give rise to broad, essentially continuous energy bands.

The thermal vibrations of the lattice are quantized, the quantized unit of acoustic energy being termed a phonon. As such this thermal broadening can also be described as a collision process, analogous to absorption or stimulated emission, in which a phonon collides with an ion leading to emission or absorption of lattice phonons. Thermal broadening in crystals is therefore often known as phonon broadening.

It is also worth noting that collisions of phonons with ions in excited states can cause the ion to decay to lower-lying levels without the emission of radiation. In non-radiative decays of this type the energy difference is carried away by lattice phonons. Non-radiative decay plays a crucial role in forming and maintaining the population inversion in many solid-state laser systems.

Finally we note that, as we might expect, both the degree of phonon broadening and the non-radiative decay rates of excited ions depend strongly on the lattice temperature.

2.4 Homogeneous broadening and the Einstein coefficients

The Einstein picture of the interaction of radiation and matter that we discussed in Lecture 1 assumed that the energy levels were perfectly sharp. It should be clear from the above that this is never so.

For a system of homogeneously broadened atoms, all atoms will interact with a beam of radiation of angular frequency $\omega$ with the same strength. However, the strength of the interaction will depend on the detuning of $\omega$ from $\omega_0$, the centre frequency of the transition.

We can account for this by incorporating the frequency dependence into spectral lineshapes:

1. The rate per unit volume at which atoms in the upper level decay to the lower level by spontaneous emission of photons with angular frequencies lying in the range $\omega$ to $\omega + \delta \omega$ is equal to $N_2 A_{21} g_A(\omega - \omega_0) \delta \omega$, where $N_2$ is the density of atoms in the upper level and $g_A(\omega - \omega_0)$ is the lineshape for spontaneous emission.

2. The rate per unit volume at which atoms in the lower level are excited to the upper level by the absorption of photons with angular frequencies lying in the range $\omega$ to $\omega + \delta \omega$ is equal to $N_1 B_{12} g_B(\omega - \omega_0) \rho(\omega) \delta \omega$, where $N_1$ is the density of atoms in the lower level, $g_B(\omega - \omega_0)$ is the lineshape for absorption, and $\rho(\omega)$ is the spectral energy density of the radiation.
3. The rate per unit volume at which atoms in the upper level decay to the lower level by stimulated emission of photons with angular frequencies lying in the range \( \omega \) to \( \omega + \delta \omega \) is equal to \( N_2 B_{21} g_B' (\omega - \omega_0) \rho(\omega) \delta \omega \), where \( g_B' (\omega - \omega_0) \) is the lineshape for stimulated emission.

Notice how, for the moment, we have allowed the frequency dependence of the three processes to be different. Notice also that the lineshapes must be normalized, which ensures that when integrated over the entire lineshape of the transition the total rates of spontaneous emission, absorption, and stimulated emission agree with the Einstein coefficients. For example, the total rate of spontaneous emission at all frequencies is given by,

\[
\int_0^\infty N_2 A_{21} g_A (\omega - \omega_0) \, d\omega = N_2 A_{21}.
\]  

(2.13)

Similar relations hold for the lineshapes for absorption and stimulated emission.

The question remains, how are the different lineshapes related? By considering the rate of emission and absorption of photons with angular frequencies lying in the range \( \omega \) to \( \omega + \delta \omega \) for a system of matter in thermal equilibrium with radiation, just as we did in Lecture 1, we may derive relations between the rates of the three fundamental processes. We find,

\[
g_{1B} B_{12} (\omega - \omega_0) = g_{2B} B_{21} g_B' (\omega - \omega_0)
\]  

(2.14)

\[
A_{21} g_A(\omega - \omega_0) = \frac{\hbar \omega^3}{\pi^2 c^3} B_{21} g_B'(\omega - \omega_0).
\]  

(2.15)

As we might expect, these are very similar to the relations that we found earlier.

We see immediately that the lineshapes for absorption and spontaneous and stimulated emission are identical; in other words the three processes all have the same frequency dependence.

But what is that frequency dependence? Well, if we were to measure the spectrum of spontaneously emitted radiation on the transition we would measure the lineshape function \( g_H (\omega - \omega_0) \). For lifetime-broadened atoms, for example, \( g_H (\omega - \omega_0) \) would be the Lorentzian function of eqn. (2.9). The measured spectrum of spontaneous emission must, therefore, give the frequency dependence of spontaneous emission, stimulated emission, and absorption.

2.5 Optical gain

We are now in a position to describe how a beam of radiation may be amplified.

Suppose that the beam propagates along the \( z \)-axis through a medium with population densities \( N_2 \) and \( N_1 \) in the upper and lower levels respectively. In general the radiation will have a finite spectral width, and is described by a spectral intensity \( I(\omega, z) \) and energy density \( \rho(\omega, z) \). We take the laser transition to be homogeneously broadened so that all atoms interact with the beam equally.

We consider the amplification of the beam as it passes through the small region lying between the planes \( z = z \) and \( z = z + \delta z \), as illustrated schematically in Fig. 2.4. As the beam passes through the medium it loses energy owing to absorption by atoms in the lower laser level, but gains energy by stimulated
emission from atoms in the upper laser level. The net rate at which atoms are transferred from the upper to the lower laser level by the stimulated emission of photons with angular frequencies lying between $\omega$ and $\omega + \delta\omega$ is,

$$\left[ N_2 B_{21} - N_1 B_{12} \right] g_H(\omega - \omega_0) \rho(\omega, z) \delta\omega \cdot A \delta z$$

(2.16)

where $A$ is the area of the beam. Each such transfer releases an energy of $\hbar \omega$ to the beam, and hence, within this frequency range, the power gained by the beam is,

$$\left[ N_2 B_{21} - N_1 B_{12} \right] g_H(\omega - \omega_0) \rho(\omega, z) \delta\omega \cdot A \delta z \cdot \hbar \omega.$$  

(2.17)

For the frequency interval under consideration, the power carried into the region by the beam is $I(\omega, z) A \delta\omega$, and hence the power gained by the beam can also be written as,

$$\left[ I(\omega, z + \delta z) - I(\omega, z) \right] A \delta\omega.$$  

(2.18)

Equating eqns. (2.17) and (2.18) we find, straightforwardly,

$$\frac{\partial I}{\partial z} = \left[ N_2 B_{21} - N_1 B_{12} \right] g_H(\omega - \omega_0) \frac{\hbar \omega}{c} I(\omega, z),$$

(2.19)

where we have used the fact that for a beam of radiation $I(\omega, z) = \rho(\omega, z) c$. This result is usually tidied in the following form:

$$\frac{\partial I}{\partial z} = N^* \sigma_{21}(\omega - \omega_0) I(\omega, z),$$

(2.20)

where we have used the relation between the Einstein B-coefficients and defined the population inversion density as,

$$N^* = N_2 - \frac{g_2}{g_1} N_1.$$  

(2.21)
and the optical gain cross-section as\(^2\),

\[
\sigma_{21}(\omega - \omega_0) = \frac{\hbar \omega_0}{c} B_{21} g_H(\omega - \omega_0)
\]

\[
\Rightarrow \sigma_{21}(\omega - \omega_0) = \frac{\pi^2 c^2}{\omega_0^2} A_{21} g_H(\omega - \omega_0)
\]

(2.23)

The advantage of writing our result in the form of eqn. (2.20) is that the right-hand side is now factored into terms which depend on (i) the population densities of the upper and lower levels; (ii) the atomic physics of the transition; (iii) the spectral intensity of the beam.

### 2.5.1 Small-signal gain coefficient

Equation (2.20) is very often re-written as,

\[
\frac{1}{I} \frac{\partial I}{\partial z} = \alpha_{21}(\omega - \omega_0)
\]

(2.24)

where \(\alpha_{21}(\omega - \omega_0)\) is known as the small-signal gain coefficient (units \(m^{-1}\)), and is given by,

\[
\alpha_{21}(\omega - \omega_0) = N^* \sigma_{21}(\omega - \omega_0).
\]

(2.25)

We will see later that it is very important to recognize that the small-signal gain coefficient is in general also a function of the beam intensity. The reason for this is that at very high intensities the increased rate of stimulated emission will reduce the population inversion, a process known as gain saturation.

For the moment, however, we will neglect that complication and assume that the population inversion is positive and independent of intensity or position. Then, integrating eqn. (2.24) we find,

\[
I(\omega, z) = I(\omega, 0) \exp[\alpha(\omega - \omega_0)z],
\]

(2.26)

We see that the beam of radiation grows exponentially with propagation distance. The reason for the energy gain of the beam is straightforward: the rate of stimulated emission from the upper level is greater than the rate of absorption from the lower level.

The small-signal gain coefficient (often loosely called ‘the gain coefficient’) is important in determining the performance of a laser system. Since the gain coefficient is proportional to the population inversion density, it is strongly dependent upon the operating conditions of the laser. For example, in a neodymium-doped yttrium aluminium garnet (Nd:YAG) laser the gain coefficient will depend on many parameters of the laser design including the strength of the optical pumping and the concentration of Nd\(^{3+}\) ions in the YAG crystal. In contrast the optical gain cross-section is a more fundamental parameter of the laser transition since it depends only on the physics of the laser transition and therefore is largely independent of the pumping conditions.

\(^2\)Once again we will set \(\omega \approx \omega_0\) in the factors multiplying the lineshape.
2.5.2 Absorption and Beer’s Law

It is worth recognizing that we can define the cross-section in various ways. The gain cross-section defined in eqn. (2.22) takes the form it does because of the way we defined the population inversion. We defined $N^* = N_2 - \frac{g_2}{g_1}N_1$ since, as laser physicists, we are predominantly interested in the population, $N_2$, in the upper level. Clearly in the ideal case the population in the lower level will be small, and then $N^* \approx N_2$.

If we studied absorption, however, we would generally be interested in the lower level population. To illustrate this, suppose that the population inversion where negative and independent of intensity or position. Equation (2.26) would then take the form,

$$I(\omega, z) = I(\omega, 0) \exp(-\kappa_{12}(\omega - \omega_0)z), \quad (2.27)$$

where the absorption coefficient is given by,

$$\kappa_{12}(\omega - \omega_0) = -(N_2 - \frac{g_2}{g_1}N_1)\sigma_{21}(\omega - \omega_0)$$

$$= -\left(\frac{g_1}{g_2}N_2 - N_1\right)\frac{g_2}{g_1}\sigma_{21}(\omega - \omega_0)$$

$$= +\left(N_1 - \frac{g_1}{g_2}N_2\right)\frac{\hbar\omega_0}{g_1}B_{21}g_H(\omega - \omega_0)$$

$$= +\left(N_1 - \frac{g_1}{g_2}N_2\right)\frac{\hbar\omega_0}{c}B_{12}g_H(\omega - \omega_0)$$

$$= +N^{**}\sigma_{12}^{\text{abs}}(\omega - \omega_0) \quad (2.28)$$

where $N^{**} = N_1 - (g_1/g_2)N_2$ and the absorption cross-section is given by,

$$\sigma_{21}^{\text{abs}}(\omega - \omega_0) = (\hbar\omega_0/c)B_{12}g_H(\omega - \omega_0). \quad (2.29)$$

Notice that the absorption cross-section follows the definition of the gain cross-section but with $B_{21}$ replaced with $B_{12}$, as we might expect.

The exponential decrease in the intensity of the beam as it propagates through the medium is known as Beer’s Law.

2.5.3 Frequency dependence of gain

In the case that $N^*$ is independent of intensity or position, eqn (2.26) describes the growth of each frequency component. Clearly the growth in intensity depends on the detuning of the frequency from the centre frequency $\omega_0$. The frequency dependence of the optical gain cross-section is simply the lineshape of the transition, $g_H(\omega - \omega_0)$. So, for example, a transition which is lifetime-broadened will have an optical gain cross-section with a Lorentzian lineshape:

$$\sigma_{21}^{\text{abs}}(\omega - \omega_0) = \frac{\hbar\omega_0}{c}B_{21}1 \frac{1}{\pi} \frac{\left(\Delta\omega_L/2\right)}{(\omega - \omega_0)^2 + \left(\Delta\omega_L/2\right)^2}, \quad (2.30)$$
where $\Delta \omega_0$ is the FWHM of the lineshape.

Clearly the optical gain cross-section will typically be strongly peaked at the transition frequency $\omega_0$. In the absence of saturation the beam grows as $\exp[N^*\sigma_{21}(\omega - \omega_0)z]$ and consequently the amplification of the beam will be an even stronger function of frequency. Hence, if a beam is input to an optical amplifier with a frequency width which is of order, or larger, than the linewidth of the transition, the spectral width of the output beam will be much narrower owing to greater amplification of frequencies close to the line centre. Frequency narrowing of this type is known as **gain narrowing**.
Lecture 3

Gain saturation

3.1 Introduction

In Lecture 2 we derived an expression for the growth in intensity of a beam propagating through an inverted, homogeneously-broadened laser medium by assuming that the population inversion density was independent of intensity or position. This result can only be true at low beam intensities since at higher intensities the increased rate of stimulated emission from the upper laser level causes the population inversion to decrease, a process known as gain saturation. In this lecture we investigate gain saturation, and by so doing will determine what we mean by ‘high’ or ‘low’ intensity.

3.2 Laser rate equations for narrow-band radiation

The bandwidth $\Delta \omega_B$ of the oscillating mode in a laser oscillator (i.e. a gain medium located within an optical cavity) will nearly always be very small compared to the spectral width of the laser transition. Likewise, for an amplifier (just a gain medium, with no optical cavity) it is also often the case that the bandwidth of the beam to be amplified is narrow compared to that of the optical transition concerned. We will therefore make this simplifying assumption in what follows.

In general the rate equation for (say) the upper laser level can then be written in the form,

$$\frac{dN_2}{dt} = R_2 - (N_2 B_{21} - N_1 B_{12}) \int_0^\infty g_H(\omega - \omega_0) \rho(\omega) d\omega + \ldots, \quad (3.1)$$

where we see that the total rate of stimulated emission is given by integrating over the lineshape, and $+\ldots$ indicate that in general other terms may appear in the rate equation. We may re-arrange eqn. (3.1) as follows:
\[
\frac{dN_2}{dt} = R_2 - \left( N_2 - \frac{g_2}{g_1} N_1 \right) \int_0^\infty \frac{h\omega}{c} B_{21}(\omega - \omega_0) \frac{\rho(\omega)c}{h\omega} d\omega + \ldots \quad (3.2)
\]

\[
= R_2 - N^* \int_0^\infty \sigma_{21}(\omega - \omega_0) \frac{I(\omega)}{h\omega} d\omega + \ldots \quad (3.3)
\]

where \( I(\omega) \) is the spectral intensity of the radiation.

For narrow-band radiation the gain cross-section varies slowly over the spectral width of the radiation and so \( I(\omega) \) acts like a Dirac delta function: \( I(\omega) = I_T \delta(\omega - \omega_L) \) where \( I_T \) is the total intensity and \( \omega_L \) the centre frequency of the beam. We then have:

\[
\frac{dN_2}{dt} = R_2 - N^* \int_0^\infty \sigma_{21}(\omega - \omega_0) \frac{I_T \delta(\omega - \omega_L)}{h\omega} d\omega + \ldots \quad (3.4)
\]

\[
= R_2 - N^* \sigma_{21}(\omega_L - \omega_0) \frac{I_T}{h\omega_L} + \ldots , \quad (3.5)
\]

Eqn. (3.5) may be interpreted as follows. We can regard \( N^* \) as the effective number density of inverted atoms, and \( \sigma_{21}(\omega_L - \omega_0) \) as their effective cross-sectional area. Since \( I_T/h\omega_L \) is the photon flux, eqn. (3.5) takes the standard form for the rate of a process in terms of a cross-section and a flux of incident particles.

### 3.2.1 Growth equation for a narrow-band beam

We have already derived the equation describing the growth of each spectral component of a beam (eqns (2.24) and (2.25)):

\[
\frac{\partial I}{\partial z} = N^* \sigma_{21}(\omega - \omega_0)I(\omega, z) . \quad (3.6)
\]

To describe the rate of growth of a beam of finite spectral width we integrate both sides of the above over the bandwidth of the beam:

\[
\int_0^\infty \frac{\partial I}{\partial z} d\omega = \int_0^\infty N^* \sigma_{21}(\omega - \omega_0)I(\omega, z) d\omega
\]

\[
\Rightarrow \frac{\partial}{\partial z} \int_0^\infty I(\omega, z) d\omega = \int_0^\infty N^* \sigma_{21}(\omega - \omega_0)I(\omega, z) d\omega
\]

\[
\Rightarrow \frac{dI_T}{dz} = \int_0^\infty N^* \sigma_{21}(\omega - \omega_0)I(\omega, z) d\omega . \quad (3.7)
\]

Again, for narrow-band radiation we may bring the cross-section outside the integral:

\[
\frac{dI_T}{dz} = N^* \sigma_{21}(\omega_L - \omega_0)I_T . \quad (3.8)
\]
3.3. Gain saturation

We are now in a position to consider how a beam of intense, narrow-band radiation (such as might be present in an operating laser) affects the population inversion produced by the pumping. Figure 3.1 shows schematically the processes which affect the level populations of a laser operating under steady-state conditions in the presence of an intense radiation beam.

Figure 3.1: Processes affecting the upper and lower laser levels in a laser operating under steady-state conditions in the presence of an intense radiation beam.

3.3 Gain saturation

We may write the rate equations for the laser levels as,

\[
\frac{dN_2}{dt} = R_2 - N^* \sigma_{21} (\omega_L - \omega_0) \frac{I}{\hbar \omega_L} \frac{N_2}{\tau_2} \tag{3.9}
\]

\[
\frac{dN_1}{dt} = R_1 + N^* \sigma_{21} (\omega_L - \omega_0) \frac{I}{\hbar \omega_L} + N_2 A_{21} - \frac{N_1}{\tau_1} \tag{3.10}
\]

where we have dropped the subscript \text{T} from the total intensity\textsuperscript{1}.

We note that:

1. We will assume that the pump rates \(R_2\) and \(R_1\) are constant and, in particular, are independent of \(N_1\) and \(N_2\);

2. The pump rates include direct excitation collision rates, indirect processes such as pumping by radiative or non-radiative cascades. However, spontaneous emission on the laser transition itself is included explicitly as \(N_2 A_{21}\).

It is straightforward to find the steady-state solutions of eqns (3.9) and (3.10):

\[
N_2 = R_2 \tau_2 - N^* \sigma_{21} \frac{I}{\hbar \omega_L} \tau_2 \tag{3.11}
\]

\[
N_1 = R_1 \tau_1 + N^* \sigma_{21} \frac{I}{\hbar \omega_L} \tau_1 + N_2 A_{21} \tau_1, \tag{3.12}
\]

\textsuperscript{1}From here onwards we will always distinguish spectral intensity by writing it in the form \(I(\omega)\).
where we have dropped the frequency dependence of $\sigma_{21}(\omega_L - \omega_0)$ to avoid clutter. Eliminating $N_2$ from eqn. (3.12) gives the population inversion density $N^* = N_2 - (g_2/g_1)N_1$ as,

$$N^* = \frac{R_2 \tau_2 [1 - (g_2/g_1)A_{21}\tau_1] - (g_2/g_1) R_1 \tau_1}{1 + \sigma_{21} \frac{I}{\hbar \omega_L} \left[ \tau_2 + (g_2/g_1)\tau_1 - (g_2/g_1)A_{21}\tau_1 \tau_2 \right]}.$$  \hspace{1cm} (3.13)

This last result looks complicated, but it is actually rather simple! Notice that the denominator equals unity when $I = 0$. Consequently, the numerator must be just the population inversion produced by the pumping in the absence of the beam. Hence we can rewrite eqn. (3.13) in the following form:

$$N^*(I) = \frac{N^*(0)}{1 + I/I_s}.$$  \hspace{1cm} (3.14)

The parameter $I_s$ is known as the saturation intensity, and is given by

$$I_s = \frac{\hbar \omega_L}{\sigma_{21}} \left( \tau_2 + \frac{g_2}{g_1} \tau_1 - \frac{g_2}{g_1} A_{21} \tau_1 \tau_2 \right)^{-1},$$  \hspace{1cm} (3.15)

which is itself usually re-written in the form,

$$I_s = \frac{\hbar \omega_L}{\sigma_{21} \tau_R},$$  \hspace{1cm} (3.16)

where $\tau_R$, the recovery time, is given by

$$\tau_R = \tau_2 + \frac{g_2}{g_1} [1 - A_{21} \tau_2].$$  \hspace{1cm} (3.17)

Let us consider eqn. (3.14) in more detail. We see that the intense beam of radiation reduces, or ‘burns down’, the population inversion by a factor of $(1 + I_T/I_s)$ as a result of the increased rate of stimulated emission from the upper laser level. The intensity of the radiation required to reduce the inversion to one-half of that achieved in the absence of the beam is the saturation intensity (units of $\text{W m}^{-2}$). For a laser oscillator or amplifier the saturation intensity is, then, a measure of the intensity to which a beam may be amplified before the increased rate of stimulated emission from the upper laser level starts to affect the level populations significantly. As such the saturation intensity marks the boundary between ‘high’ and ‘low’ intensity.

### 3.3.1 Approximations for the saturation intensity

It is often possible to find different approximations for the saturation intensity. For example, in so-called ‘four-level’ lasers the fluorescence lifetime of the lower level is much shorter than that of the upper level. In this special case $\tau_R \approx \tau_2$. The same approximation for the recovery time occurs when the upper laser
level decays predominantly by radiative decay on the laser transition itself, whereupon \( \tau_2 \approx A_{21}^{-1} \). For these special cases we find,

\[
I_s \approx \frac{\hbar \omega_i}{\sigma_{21} \tau_2} \quad \text{Special case if } \frac{\tau_2}{\tau_1} \gg 1 \text{ or } A_{21} \tau_2 \approx 1. \tag{3.18}
\]

### 3.3.2 Saturated gain coefficient

Having found the burnt down population inversion density, the saturated gain coefficient is found from,

\[
\alpha_I(\omega - \omega_0) = N^*(I)\sigma_{21}(\omega - \omega_0), \tag{3.19}
\]

where the subscript \( I \) indicates that \( \alpha \) is now a function of the beam intensity.

Hence,

\[
\alpha_I(\omega - \omega_0) = \frac{\alpha_0(\omega - \omega_0)}{1 + I/I_s}, \quad \text{Saturated gain coefficient} \tag{3.20}
\]

where \( \alpha_0(\omega - \omega_0) \) is the small-signal gain coefficient, i.e. the gain coefficient experienced by, or measured with, a beam of intensity much less than the saturation intensity.

### 3.4 Beam growth in a laser amplifier

From the analysis above we know that a narrow-band beam of radiation will grow as it propagates through a length of inverted medium according to:

\[
\frac{dI}{dz} = \alpha_I I = \frac{\alpha_0}{1 + I/I_s} I, \tag{3.21}
\]

which may be integrated to give,

\[
\ln \left[ \frac{I(z)}{I(0)} \right] + \frac{I(z) - I(0)}{I_s} = \alpha_0 z. \tag{3.22}
\]

This transcendental equation has algebraic solutions in two limits:

\[
I(z) = I(0) \exp(\alpha_0 z); \quad \text{Weak beam } (I(z) \ll I_s) \tag{3.23}
\]
\[
I(z) = I(0) + \alpha_0 I_s z; \quad \text{Heavy saturation } (I(0) \gg I_s) \tag{3.24}
\]

In other words, at low intensities the intensity of the beam grows exponentially with distance, just as we found earlier; whereas once the laser transition becomes heavily saturated the intensity grows linearly.
3.5 Cavity effects

A simple laser amplifier consists of one or more sections of inverted media such that a beam injected at one end is amplified as it propagates. In a laser oscillator the gain medium is located within an optical cavity, such that the laser radiation originates within the gain medium itself and is amplified as it circulates around the optical cavity. Unless qualified, the term ‘laser’ usually refers to a laser oscillator. In this section we examine the effect of the optical cavity on the operation and output of a laser oscillator.

Let us first consider a simple laser cavity comprising two mirrors separated by a distance \( L_c \), as illustrated in Fig. 3.2. For the moment we will consider the case in which there is no gain medium within the cavity.

Suppose that radiation is circulating within the cavity. Under steady-state conditions we can represent the radiation by a wave propagating towards positive \( z \), denoted by a subscript \( + \), and a wave propagating to negative \( z \) denoted by a subscript \( - \).

The positive- and negative-going waves will be one (or a superposition) of the transverse modes of the cavity. The transverse modes of the cavity are eigen-modes of the cavity which, when launched once round the cavity return with the same transverse spatial variation. As such they depend on the curvature of the cavity mirrors and their location. For the present purpose we need not concern ourselves with the detailed form of the transverse modes and we merely denote the modes of the positive- and negative-going waves by the functions \( u_\pm(\mathbf{r}, \omega, t) \). However, in anticipation of the introduction of gain into the cavity, we will allow the overall amplitude of the positive- and negative-going waves to vary with position along the cavity axis according to \( a_\pm(z, \omega) \).

Let us now consider the amplitude of the radiation field immediately to the right of mirror 1, i.e. at \( z = 0 \). Under steady-state conditions, after reflection from mirror 1 the negative-going wave must be the same as the right-going wave at \( z = 0 \). Hence we have:
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\[ a_+(0, \omega)u_+(r, \omega, t) = a_- (0, \omega) r_1 e^{i\phi_1} u_-(r, \omega, t). \]  

(3.25)

where \( r_1 \) and \( \phi_1 \) are the modulus and phase respectively of the amplitude reflection coefficient of mirror 1. This condition must hold at all points in the plane \( z = 0 \), and so in this plane we must have \( u_+(r, \omega, t) = u_-(r, \omega, t) \), that is that the positive- and negative-going waves must be in the same mode. We can therefore simplify eqn (3.25) to,

\[ a_+(0, \omega) = a_- (0, \omega) r_1 e^{i\phi_1}. \]  

(3.26)

Similarly, after propagating to mirror 2 and back, the right-going wave starting at \( z = 0 \) must be the same as the negative-going wave at \( z = 0 \):

\[ a_-(0, \omega) = a_+ (0, \omega) e^{i(\delta_{rt} - \phi_1)} r_2, \]  

(3.27)

where \( \delta_{rt} \) is the optical phase accumulated by the mode in propagating one complete round trip (including any phase shifts arising from the two mirrors), and \( r_2 \) and \( \phi_2 \) are the modulus and phase respectively of the amplitude reflection coefficient of mirror 2. We can combine this result with eqn (3.26) to give,

\[ a_+(0, \omega) = a_+(0, \omega) r_2 r_1 e^{i\delta_{rt}}. \]  

(3.28)

It must therefore be that under steady-state conditions,

\[ r_2 r_1 e^{i\delta_{rt}} = 1. \]  

(3.29)

Equation (3.29) is complex and hence we can extract 2 conditions. The imaginary part of the right-hand side is zero, which means that \( \delta_{rt} \) must equal an integer times \( \pi \). Further, \( r_1, r_2 > 0 \) and hence \( \exp(i\delta_{rt}) \) must also be positive. Thus we conclude that eqn (3.29) can only be satisfied if

\[ \delta_{rt} = 2\pi p. \quad p = 0, 1, 2, 3, \ldots \]  

(3.30)

Of course, (3.29) also requires \( r_1 = r_2 = 1 \) which would require perfect mirrors. The fact that in a real cavity \( |r_1|, |r_2| < 1 \) means that steady-state conditions can only be achieved if we introduce gain into the cavity, as we discuss in the following section.

Let us consider the condition on \( \phi_{rt} \) in more detail by way of a simple example. Suppose that the cavity is filled uniformly with material of refractive index \( n \). Then, \( \phi_{rt} = 2kL_c \) and hence we could write eqn (3.30) as

\[ 2k_p L_c = 2\pi p \quad p = 0, 1, 2, 3, \ldots. \]  

(3.31)

In terms of frequency (not angular) this condition becomes:
We see that only certain frequencies will oscillate, corresponding to the resonant frequencies or longitudinal modes of the optical cavity. The frequency spacing between adjacent longitudinal modes is
\[ \Delta \nu_{p,p-1} = \frac{c}{2nL_c}. \]  

3.5.1 Insertion of gain into the cavity: The laser threshold condition

Now let us suppose that we place a gain medium of length \( L_g \) within the optical cavity. Now as radiation circulates around the cavity it will be amplified as it passes through the gain medium, and so may overcome the cavity losses.

We wish to calculate a condition for the laser reaching the threshold for laser oscillation. Near threshold the intensity of the circulating radiation must be very small, so that the gain will be unsaturated, and the circulating radiation is amplified according to the unsaturated gain coefficient. Denoting the threshold value of the unsaturated gain coefficient by \( \alpha_{th}^0 \), with the insertion of gain in the optical cavity eqn (3.27) becomes at threshold:

\[
a_-(0, \omega) = a_+(0, \omega) \exp \left( \frac{1}{2} \alpha_{th}^0(\omega)L_g \right) \exp \left[ -\frac{1}{2} \kappa(\omega)L_c \right] \times r_2 \exp \left( \frac{1}{2} \alpha_{th}^0(\omega)L_g \right) \exp \left[ -\frac{1}{2} \kappa(\omega)L_c \right] \exp (i(\delta_{rt} - \phi_1)) \tag{3.34} \]

Here \( \kappa \) is an absorption coefficient, which is introduced to represent any (unwanted) losses within the cavity in addition to the losses at the cavity mirrors. For convenience we have assumed that these losses are distributed uniformly over the whole length, \( L_c \), of the cavity. Notice how, for example, the amplitude of the positive-going wave is amplified by a factor of \( \exp \left( \frac{1}{2} \alpha_{th}^0(\omega)L_g \right) \) — the factor of \( \frac{1}{2} \) arises since we are calculating the growth of the amplitude of the radiation rather than its intensity. The same factor appears in the term representing absorption for the same reason.

Combining eqns (3.34) and (3.26) we have

\[
r_2 r_1 \exp \left[ \alpha_{th}^0(\omega)L_g \right] \exp [-\kappa(\omega)L_c] \exp (i\delta_{rt}) = 1. \tag{3.35} \]

Following the argument above, we see that we must have \( \exp (i\phi_{rt}) = 1 \), and hence the frequency of the radiation must correspond to one of the cavity modes.

Equating the real parts of both sides of eqn (3.35) gives,

You should convince yourself that this is just the free-spectral range of the cavity when used as a Fabry-Perot etalon.
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\[ r_1 r_2 \exp[\alpha^{th}_0(\omega_p)L_g] \exp[-\kappa(\omega_p)L_c] = 1, \]  
\[ \text{(3.36)} \]

where the subscript \( p \) reminds us that the frequency must correspond to a cavity mode. Taking the square-modulus of both sides we find the threshold condition for lasing in terms of the power reflectivities \( R_1 \) and \( R_2 \) of the cavity mirrors:

\[ R_1 R_2 \exp[2\alpha^{th}_0(\omega_p)L_g] \exp[-2\kappa(\omega_p)L_c] = 1. \]  
\[ \text{Threshold condition (3.37)} \]

This last result has a straightforward interpretation: if we imagine launching a beam of unit intensity from within the cavity, after one round-trip it would have an intensity equal to the left-hand side of eqn (3.37); if this is less than unity the beam will decay in intensity with each round-trip; if it is greater than unity, the beam will grow in intensity. Equation (3.37) therefore gives the condition for steady-state laser oscillation. It may be written in the alternative form,

\[ 2\alpha^{th}_0(\omega_p)L_g = 2\kappa(\omega_p)L_c - \ln(R_1 R_2). \]  
\[ \text{Threshold condition (3.38)} \]

To summarize, laser oscillation can only occur if for some cavity mode the unsaturated round-trip gain exceeds the round-trip loss. The threshold condition determines — for a given laser cavity — a threshold value for the gain coefficient. In turn this defines, for a given laser transition, threshold values for the population inversion and the rate of pumping of the upper laser level.

3.5.2 Operation above threshold

In deriving the conditions required to reach laser oscillation we noted that under steady-state conditions the round-trip gain experienced by an oscillating cavity mode must be balanced by the round-trip loss. Of course, this must also be true if the laser is operating above threshold — that is, we are pumping the laser gain medium harder. The important difference in this case, however, is that the gain experienced by the radiation circulating within the cavity will be saturated, at least to some extent, by the laser radiation circulating within the cavity. In the section below we shall see how it is this saturation that reduces the round-trip gain so that it is in balance with the cavity losses as the pumping of the laser levels is increased to ever higher values.

In the special case that the saturated round-trip gain is small, we may write the condition for steady-state operation of a laser operating above threshold as,

\[ R_1 R_2 \exp[2\alpha_I(\omega_p)L_g] \exp[-2\kappa(\omega_p)L_c] = 1 \]  
\[ \text{(3.39)} \]

where \( \alpha_I(\omega_p) \) is the saturated gain coefficient at the frequency \( \omega_p \) of the oscillating mode. This result only applies if the saturated round-trip gain is small or, equivalently, if the round-trip losses are small — such as would be the case if the absorption is negligible and the mirrors have high reflectivities.
Equation (3.39) is not valid if the round-trip gain is large, since then the intensity of a beam of radiation passing once through the gain medium is not given by $\exp[\alpha_I(\omega_p)L_g]$ but must be found by integrating equations of form of eqn (3.21). However, we emphasize that even in these more complicated situations, in steady-state conditions the round-trip gain is balanced by the round-trip loss.

### 3.6 Laser operation above threshold

We now consider how a homogeneously-broadened laser system behaves as we gradually increase the pumping of the gain medium (by, for example, increasing the flashlamp voltage). As the pumping is increased from below the threshold value the population inversion $N^*$ will increase until the small-signal gain for the mode closest to the line centre reaches the threshold value determined by the cavity losses. This will cause a very large (of order $10^{15}$) increase in the energy density of the radiation field at the frequency of the oscillating mode.

What happens if the pumping is increased further? Perhaps surprisingly, neither the gain coefficient $\alpha_I(\omega)$, or the population inversion will increase. The reason is that in the steady-state, the round-trip gain must always be balanced by the round-trip loss.

How is this balance maintained? Whilst it is true that increasing the pumping rate $R_2$ will increase the rate at which the upper laser level is populated, this is accompanied by an increase in the intensity of the oscillating mode which increases the rate of depopulation of the upper laser level by stimulated emission. Thus, the increased radiation intensity acts to burn down the population inversion and maintains it (and the gain coefficient) so that the round-trip gain equals the round-trip loss. It is important to note that, since the gain medium is homogeneously-broadened, the entire population inversion is burnt down to the threshold value so that the entire gain profile $\alpha(\omega)$ is locked at the threshold value so that $\alpha_I(\omega_p) = \alpha_{th}^0(\omega_p)$.

This process is shown schematically in Figure 3.3.

### 3.6.1 Spatial hole-burning

It would seem from this argument that only one mode could ever be brought into oscillation in a homogeneously broadened laser. This is not always so, however. Within a laser cavity the oscillating mode forms a standing wave. Near the nodes of the standing wave the electric field is at all times small, and so the population inversion in this region will not be burnt down to the same extent as near the anti-nodes. Consequently a longitudinal mode at a slightly different frequency can establish itself by feeding on the high levels of the population inversion found at the nodes, as illustrated schematically in Fig. 3.4. This process is known as spatial hole-burning. Spatial hole-burning can be avoided by using

---

3. The behaviour is quite different for inhomogeneously-broadened gain media.

4. This last statement is a slight simplification (which is perfectly adequate for the present purpose). If the intensity varies significantly from one end of the gain medium to the other, as would happen if the cavity losses were very high, we cannot talk about ‘the’ gain coefficient since the gain coefficient will depend on $z$. However, even in these more complicated situations it remains the case that saturation will burn the population down to the point that the round-trip gain is balanced by the round-trip loss.
Figure 3.3: Schematic diagram showing, for a homogeneously-broadened laser transition, the effect of increasing the pump rate above threshold on: (a) the gain coefficient; (b) the intensity of the oscillating mode. Note that the gain curve does not change with increased pumping, as explained in the text.

A ring cavity⁵ and restricting laser oscillation to one direction round the cavity, since then no standing wave is formed.

---

⁵A laser cavity formed from three or more mirrors in which the path taken by the light forms a polygon of finite area.
Figure 3.4: Spatial hole-burning in a homogeneously broadened steady-state laser oscillator. In (a) the standing wave of an oscillating mode causes the gain to be depleted where the intensity is high. However, the gain is unaffected near the nodes of the standing wave since the intensity in these regions is low. This can allow other cavity modes — which will have a different frequency and hence different locations for the nodes and antinodes in its standing wave pattern — to feed off the regions of unused gain, as shown in (b).
4.1 Output power of a low-gain CW laser

In the last lecture we investigated the behaviour of a continuously-operating, or continuous wave (CW), homogeneously-broadened laser as the pumping was raised above the threshold for laser oscillation. We now investigate the operation of the laser as the pumping is raised well above threshold, and in particular examine the conditions which optimize the output power of the laser. The problem is illustrated schematically in Fig. 4.1.

As we discussed earlier, under steady-state conditions the round-trip gain must be balanced by the round-trip losses. Provided that the cavity losses are small, this condition may be written as (eqn (3.39)):

\[ 2\alpha L_c = 2\kappa L_g - \ln(R_1 R_2). \]

We will restrict our discussion to the case of low-gain lasers for which both sides of the above equation are relatively small. Imagine launching a beam of unit intensity round the cavity. Ignoring any losses, the intensity after one round-trip would be \( \exp(2\alpha L_g) \), and hence the fractional gain is...
exp(2αILg) − 1. For a low-gain system the fractional gain is therefore,

\[ \delta_{\text{gain}} = \exp(2\alpha ILg) - 1 \approx 2\alpha ILg. \]  

(4.2)

Similarly the fractional absorption is given by,

\[ \delta_{\text{abs}} = 1 - \exp(-2\kappa ILc) \approx 2\kappa ILc. \]  

(4.3)

The optical cavity will also have losses associated with the finite reflectivity of the mirrors. For each mirror we may write,

\[ R + T + A = 1, \]  

(4.4)

where \( T \) is the transmittance and \( A \) the absorptance.

For a low-gain laser \( R_1 \) and \( R_1 \) must both be close to unity and hence,

\[
\ln(R_1 R_2) = \ln[(1 - T_1 - A_1)(1 - T_2 - A_2)] \\
\approx \ln(1 - T_1 - A_1 - T_2 - A_2) \\
\approx -T_1 - A_1 - T_2 - A_2.
\]  

(4.5)

Taking the mirror through which we extract power to be mirror 2, \( T_2 \) represents the useful part of the cavity losses. The remaining terms in the above are unwanted absorption or transmission through the rear cavity mirror. We may now rewrite eqn. (4.1) in the form,

\[ 2\alpha ILg = \delta_{\text{loss}} + T_2, \]  

Low-loss threshold condition  

(4.6)

where the fractional loss is given by,

\[ \delta_{\text{loss}} = \delta_{\text{abs}} + T_1 + A_1 + A_2. \]  

(4.7)

The gain coefficient appearing on the left-hand side of eqn. (4.6) is the saturated gain coefficient:

\[ \alpha_I = \frac{\alpha_0}{1 + I/I_s}. \]  

(4.8)

Now, in a laser cavity we have two beams: one travelling to the left, and one to the right, and hence the total intensity experienced at any point in the gain medium may be written \( I(z) = I_+(z) + I_-(z) \). Since we are considering low-gain systems the intensity of the right- and left-going beams will not vary strongly with position, and the two beams will have almost the same intensity: \( I_+(z) \approx I_-(z) \). Hence, we may write \( I = I_+ + I_- \approx 2I_+ \).

The condition for steady-state oscillation then becomes,

\[ 2\alpha_0 L_g = \delta_{\text{loss}} + T_2, \]  

(4.9)
from which we find the intensity of the right-going beam to be,

\[ I_+ = \frac{1}{2} I_s \left( \frac{2\alpha_0 L_g}{T_2 + \delta_{\text{loss}}} - 1 \right). \] (4.10)

The power coupled out of the cavity is simply \( T_2 I_+ A_{\text{mode}} \), where \( A_{\text{mode}} \) is the cross-sectional area of the beam. Hence the output power of the laser is,

\[ P = \frac{1}{2} I_s A_{\text{mode}} T_2 \left( \frac{2\alpha_0 L_g}{T_2 + \delta_{\text{loss}}} - 1 \right). \] (4.11)

The general form of the variation of output power with \( T_2 \) (in this context usually referred to as the ‘output coupling’) is illustrated schematically in Fig. 4.2. The variation may be explained as follows:

- At large values of \( T_2 \) the losses of the cavity are larger than the maximum possible round-trip gain and consequently no laser oscillation occurs.

- As \( T_2 \) is decreased the threshold for laser oscillation is reached when \( T_2^{\text{thresh}} = 2\alpha_0 L_g - \delta_{\text{loss}} \).

- As \( T_2 \) is decreased below the threshold value the lower cavity losses mean that the required round-trip gain also decreases. This is achieved by an increase in the intra-cavity intensity. For \( T_2 \) below the threshold value the intra-cavity intensity always increases as \( T_2 \) is decreased.

- As \( T_2 \) is decreased below the threshold value the increase in intra-cavity intensity causes the output power to increase. Eventually, however, the decrease in the fraction of power coupled out through \( T_2 \) outweighs the increase in intra-cavity intensity and the output power decreases. Clearly the output power will be zero when \( T_2 = 0 \), although the intra-cavity intensity will be high.

Finally we note that the output power varies linearly with the small-signal gain coefficient \( \alpha_0 \). The small-signal gain is proportional to the unsaturated population inversion, and this in turn will often vary approximately linearly with the pump rate \( R_2 \) of the upper level. As a consequence, the output power of a laser operating above threshold often increases linearly with the pumping (e.g. with the discharge current, the diode current, the flashlamp energy etc.)

### 4.2 Solid-state lasers

A very large number of scientifically and commercially important lasers operate with solid-state gain media. Solid-state lasers, as distinct from lasers operating in gases or liquids, are attractive since they can often be made to be rugged and compact, there are no gas or liquid bottles with their associated handling equipment, and there is no chemical hazard. Of course, solid-state lasers are restricted to operating in wavelength ranges with good transmission through solid materials (i.e. infrared to ultraviolet). In addition, the time-averaged
output power that a solid-state laser can deliver is often limited by the need to remove waste heat from the gain medium so as to avoid thermal distortion or even melting.

For many solid-state lasers the active lasant species are impurity ions doped into an insulating crystalline or glassy solid. Figure 4.3 shows schematically the energy level diagram of the active laser ions in such a system.

It is worth noting at this point that the spectroscopy of impurity ions doped into solid hosts is in general complex owing to the large interactions with neighbouring ions. This interaction is often described in terms of a crystalline electric field which is in general non-isotropic and, for crystalline hosts, has a symmetry determined by the underlying crystal structure. As a result, the structure and labelling of the energy levels can be very different from that of the isolated ion. Further, the active ions will interact with lattice vibrations (phonons) within the solid. Such interactions can lead to substantial line broadening and rapid non-radiative decay by stimulated or spontaneous emission of phonons. Phonon broadening of levels that would be distinct in an isolated ion can form broad absorption bands.

The basic operation of this type of laser is as follows:

- The population inversion is achieved by optical pumping on broad pump bands $0 \rightarrow 3$. The broad nature of the pump bands reduces the difficulty of carefully matching the frequency of the pump light to that of the pump transition, and also allows pumping by broad-band sources such as flashlamps.

- Level 3 decays rapidly by non-radiative processes to populate a range of lower levels. The proportion of decays from level 3 that populate the upper laser level 2 is known as the branching ratio $\eta_{\text{branch}}$.

- If the pumping is sufficient to realize a population inversion, lasing occurs on the transition $2 \rightarrow 1$.  

Figure 4.2: Output power vs. the transmission $T_2$ of the output coupler for a low-gain laser system (eqn. (4.11)).
4.3. THE RUBY LASER

If the lower laser level lies close to \( E_1 \ll k_B T \), or is, the ground state it will have a large thermal population. In this case there are essentially only 3 levels of importance, and the laser is classed as a **three-level laser**.

In contrast, if the lower level is well above \( E_1 \gg k_B T \) the ground state, the laser is classed as a **four-level laser**. To avoid build-up of population in the lower laser level, known as **bottlenecking**, it is desirable for the lifetime of the lower level to be short.

In the sections below we describe an example of each of these two classes of laser system.

### 4.3 The ruby laser

In 1960 Maiman demonstrated laser oscillation in the optical region of the spectrum for the first time, using as the active medium a crystal of ruby. The active ion in ruby is \( \text{Cr}^{3+} \) doped at a level of around 0.05% by weight into sapphire (Al\(_2\)O\(_3\)). The energy levels of interest to the ruby laser are shown schematically in Fig. 4.4.

The \( \text{Cr}^{3+} \) ion has broad energy bands, \( ^4F_1 \) and \( ^4F_2 \) (in more modern notation often labelled \( ^4T_1 \) and \( ^4T_2 \)), which lie approximately \( 24 \times 10^3 \) and \( 18 \times 10^3 \) cm\(^{-1}\) above the \( ^4A_2 \) ground state\(^1\). These absorption bands in the green and violet part of the visible spectrum cause a ruby laser rod to appear pink. Ruby gemstones contain a much higher concentration of \( \text{Cr}^{3+} \), nearer 1%, and consequently are a rich red colour.

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\(^1\)The labels of the energy levels are not LS notation, or similar, but relate to symmetry properties.
Non-radiative relaxation of the \(^4F\) levels to the \(^2E\) levels is very fast (\(\tau \approx 50\) ns), and more probable than radiative decay back to the ground state. Optical pumping on the \(^4F \rightarrow ^4A_2\) pump bands therefore results in efficient transfer of population from the ground state to the upper laser levels.

The upper laser levels decay predominantly by emission of radiation on transitions to the ground state. However, the radiative decay is electric dipole forbidden, and consequently the lifetime of the upper levels is long (\(\tau \approx 3\) ms). The long lifetime of the upper level means that it can act as a ‘storage’ level, which helps the formation of a population inversion.

In ruby there are two upper laser levels, \(^2A\) and \(^E\), separated by 29 cm\(^{-1}\). The relative populations of these two levels is essentially thermalized by non-radiative transitions, and since \(k_B T \approx 200\) cm\(^{-1}\) at room temperature the population of the \(^2A\) level is approximately \(\exp(-29/210) \approx 87\%\) of that of the \(^E\) level. Laser oscillation therefore usually occurs on the \(R_1\) transition at 694 nm, although lasing on the \(R_2\) transition at 693 nm is possible with a frequency-selective cavity.

We see that the energy level structure of ruby has some desirable features for a laser system: a strong, selective mechanism for pumping the upper laser level; a relatively long upper level lifetime which helps the population inversion build-up. The major drawback, however, is that the lower laser level is the ground state! As such it will have a very large, non-decaying population. The only way that lasing can occur in ruby is if a large proportion (essentially half) of the ground state population can be transferred to the upper laser levels. It is interesting to estimate the pump energy required to do this.

### 4.3.1 Threshold for pulsed operation

Figure 4.5 shows a simplified energy level diagram for calculating the pump energy to reach the threshold for laser oscillation in ruby. Note that for simplicity we will treat the two upper laser levels as a single level with a degeneracy \(g_2 = 4\), equal to that of the ground state.
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Assuming that the cavity losses are small the threshold condition for laser oscillation may be written,

\[ 2\alpha_0 L_g = \delta_{\text{loss}} + T_2 \quad (4.12) \]
\[ 2\sigma_{21} N^{*}_{\text{thresh}} L_g = \delta_{\text{loss}} + T_2, \quad (4.13) \]

where \( N^{*}_{\text{thresh}} \) is the threshold population inversion.

The ruby laser is clearly a three-level system, and since level 3 decays so rapidly to the upper laser level we have \( N_3 \approx 0 \). We may then write,

\[ N_T = N_2 + N_1 \quad (4.14) \]
\[ N^{*}_{\text{thresh}} = N^{\text{thresh}}_2 - N^{\text{thresh}}_1, \quad (4.15) \]

where \( N_T \) is the total \( \text{Cr}^{3+} \) ion density, and \( N^{\text{thresh}}_2 \) and \( N^{\text{thresh}}_1 \) the upper and lower population densities at threshold. We then find, straightforwardly that,

\[ N^{\text{thresh}}_2 = \frac{N_T + N^{*}_{\text{thresh}}}{2}. \quad (4.16) \]

In practice, however, the threshold population inversion density required for lasing will be tiny compared to the total ion density, and hence

\[ N^{\text{thresh}}_2 = \frac{N_T}{2}. \quad (4.17) \]

In other words, as hinted above, the main impediment to laser oscillation is the huge population density in the ground state. Once the ground state has been
sufficiently depleted to equalize the populations in the upper and lower laser levels, the extra inversion required to overcome the cavity losses is relatively small.

Having calculated the threshold upper level population density, it is straightforward to estimate the energy required to do this. Taking the pump laser radiation to have a wavelength of \( \lambda_p \approx 500 \text{ nm} \), the energy required to raise each ion to the upper laser level is \( \frac{hc}{\lambda} \approx 2.5 \text{ eV} \). The threshold pump energy is therefore,

\[
E_{\text{thresh}}^{\text{abs}} = \frac{N_T}{2} \pi a^2 L_g \frac{hc}{\lambda_p},
\]

(4.18)

where \( a \) and \( L_g \) are the radius and length of the laser rod. Taking \( a = 5 \text{ mm} \), \( L_g = 20 \text{ mm} \), and \( N_T = 2 \times 10^{19} \text{ cm}^{-3} \), we find \( E_{\text{thresh}}^{\text{abs}} \approx 6 \text{ J} \).

This last figure, however, is the energy that must be absorbed by the laser rod. The electrical energy supplied must be larger by a factor of about 60 as follows:

\times 2 \text{ To achieve uniform pumping within the laser rod the doping and diameter must be such that only approximately 50\% of the pump photons are absorbed. If the Cr}^{3+} \text{ concentration is higher, or the rod diameter larger, population inversion is only achieved near the surface of the rod;}

\times 8 \text{ Only approximately 12\% of the output of the flashlamp will lie in the pump bands;}

\times 2 \text{ Only about 50\% of the pump light is geometrically coupled into the rod;}

\times 1.7 \text{ Only 60\% of the electrical energy is converted into light.}

In our numerical example, the threshold pump energy that must be supplied to the flashlamps is therefore of order 360 J.

### 4.3.2 Threshold for CW operation

In order to calculate the pump power required to achieve continuous operation of the ruby laser, we note that the upper level population density will decay spontaneously at a rate \( N_2/\tau_2 \). Consequently the threshold pump power for CW operation is simply the threshold pump energy divided by the upper laser level lifetime. With \( \tau_2 \approx 3 \text{ ms} \) we find a threshold electrical pump power of order 100 kW! The required pump power can be decreased by reducing the volume of the laser rod, but continuous operation of the ruby laser is still technically very difficult and because of this CW operation is of little practical importance.

### 4.3.3 Practical devices

Figure 4.6 shows the general layout employed for flashlamp-pumping a solid-state laser. A key component is the pumping chamber used to couple light from the flashlamp (or lamps) into the laser rod. A wide variety of geometries for the pump chamber have been developed, including double-elliptical cavities for coupling light from two flashlamps into a single laser rod, and cavities employing
4.4. THE ND:YAG LASER

Figure 4.6: Schematic diagram showing: (a) the general layout of a flashlamp-pumped solid-state laser; (b) use of an elliptical pumping chamber to increase the efficiency with which flashlamp radiation is coupled into the laser rod.

Figure 4.7: Schematic diagram of the construction of a ruby laser.

highly-reflecting diffusive surfaces which can give very uniform illumination of the active medium. The flashlamp can also be coiled around the laser rod — just as in old sci-fi movies — which avoids the need for a cavity for the pump light but is less efficient.

Figure 4.7 shows the construction of one design of practical ruby laser. Note that in this design pump chamber is ellipsoidal, rather than the cylinder of elliptical cross-section shown in Fig 4.6. Notice also that the laser cavity is formed by coating the two ends of the ruby rod with silver to provide essentially 100% reflection from the left-hand end of the rod, and partial transmission through the right-hand end.

4.4 The Nd:YAG laser

The most widely used laser based on doping of the active ion in an insulating solid is the Nd:YAG laser in which Nd$^{3+}$ ions are doped into an Yttrium
Aluminium Garnet (YAG) crystalline host.

A simplified energy level diagram for Nd$^{3+}$ ions doped in YAG is shown in Fig. 4.8. The ground configuration of the Nd$^{3+}$ ion comprises a Xe-like core plus three 4f electrons. All of the levels of interest arise from this configuration owing to differing interactions of the 4f electrons with other electrons in the ion and Stark effects in the crystal field. The Nd$^{3+}$ ions occupy equivalent sites in the YAG lattice, and hence the centre frequencies of the transitions are the same for all ions. The laser lines are broadened by phonon interactions to a width of approximately $2 \times 10^{11}$ Hz.

The laser pump bands occur at wavelengths of approximately 730 nm and 800 nm, and in fact this absorption bands cause the otherwise colourless YAG crystal to appear purple. Ions excited to these levels decay rapidly, and non-radiatively, with a lifetime of order 100 ns to the $^4F_{3/2}$ level at approximately 11500 cm$^{-1}$. There is a significant energy gap from the $^4F_{3/2}$ level to the next lower level, and consequently the rate of non-radiative decay of the upper laser level is relatively slow. Instead the level decays predominantly by radiative emission to the $^4I$ levels, with a lifetime of 230 µs. The strongest transition is to the lower laser level, $^4I_{11/2}$, at approximately 2000 cm$^{-1}$. The wavelength of the Nd:YAG laser is therefore $1064$ nm. We should also note that the $^4F_{3/2}$ and $^4I_{11/2}$ levels are split into 2 and 6 levels respectively, and in fact lasing occurs on two, closely-spaced transitions.

Since the lower laser level lies well above the ground state, its thermal population will be small relative to that of the ground state (a fraction of approximately $\exp(-2000/210) \approx 10^{-4}$). Further, the lower laser level undergoes rapid (ns) non-radiative decay to the ground state, and consequently the population of the lower laser level is small even during lasing. As such the Nd:YAG laser is

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2This is not the case for Nd$^{3+}$ ions doped into glass, and consequently Nd:Glass lasers are inhomogeneously-broadened and have a much larger linewidth.

3The use of non-linear crystals to frequency double, triple, or even quadruple the fundamental of the Nd:YAG laser to wavelengths of 532 nm, 354 nm, and 266 nm is very common in applications of this laser.
4.4. THE ND:YAG LASER

4.4.1 Threshold for pulsed operation

Figure 4.9 shows an idealized energy level diagram for calculation of the threshold pump energy for laser oscillation.

Again, the threshold population inversion is given by eqn. (4.13). Taking the losses on the right-hand side to be (say) 10%, and assuming a typical length of laser rod of $L_g = 50$ mm and a gain cross-section of $6 \times 10^{-19}$ cm$^2$, we find $N_{\text{thresh}}^* \approx 1.6 \times 10^{16}$ cm$^{-3}$. Now since $N_1 \approx 0$ we have, simply,

$$N_{2,\text{thresh}}^* = N_{\text{thresh}}^*$$  \hspace{1cm} \text{Threshold condition for YAG:} \hspace{1cm} (4.19)$$

Notice that this is three orders of magnitude smaller than the threshold upper level population density of $N_f/2 \approx 1 \times 10^{19}$ cm$^{-3}$ required to achieve laser oscillation in ruby. The reason for the much lower threshold energy is straightforward: for Nd:YAG essentially any excitation to the upper laser level constitutes a population inversion, and we then only need to overcome the cavity losses. In contrast, for the ruby laser we have to excite half the total ion density before the population inversion even starts to be positive.

Having estimated $N_{2,\text{thresh}}^*$ calculation of the threshold pump energy proceeds as before. Taking $\lambda_p = 800$ nm and $a = 2$ mm we find $E_{\text{abs}} \approx 2.5$ mJ. The same conversion efficiencies that we used for ruby gives a threshold electrical pulse energy of order 150 mJ.
4.4.2 Threshold for CW operation

The power absorbed for continuous operation is given by dividing by $\tau_2$, which for our numerical example yields an absorbed power of 11 W and a total input electrical power of order 650 W. These parameters can be achieved easily.

4.4.3 Practical devices

Flashlamp pumped Nd:YAG lasers have a similar geometry to that of the Ruby laser although, of course, both pulsed and CW versions of the Nd:YAG laser are available.

The Nd:YAG laser can also be optically pumped with the radiation from high-power semiconductor, or ‘diode’, lasers and in particular GaAlAs lasers operating close to the 800 nm pump band of Nd:YAG. Diode lasers of this type are very efficient, their electrical efficiency can reach 50%. Pumping with diode lasers has several advantages over flashlamp pumping: all of the output lies within the pump band of the Nd:YAG laser, improving efficiency and reducing thermal load on the system; the output is unidirectional and so easier to couple in to the laser rod. As a result the overall electrical efficiency of diode-pumped Nd:YAG (and other diode-pumped solid-state) lasers can exceed 1%.
5

Problems

1. Stimulated and spontaneous transitions: A blob of matter is placed in a cavity and allowed to interact with blackbody radiation of temperature $T$.

(a) Show that for a transition of angular frequency $\omega_{21}$, the rate of stimulated emission becomes equal to that of spontaneous emission when,

$$k_B T = \frac{\hbar \omega_{21}}{\ln 2}$$

(b) Calculate this temperature for the following transitions:
   i. radio frequencies of 50 MHz
   ii. microwaves at 1 GHz
   iii. visible light of wavelength 500 nm
   iv. X-rays of energy 1 keV

(c) Show that total rate of radiative transitions from an excited level to a lower level, i.e. the sum of the spontaneous and stimulated rates, can be written as:

$$R_{\text{rad}}^2 = N_2 A_{21} \left[ \frac{\pi^2 c^3}{\hbar^3 \omega_{21}} \rho(\omega_{21}) + 1 \right]$$

(d) Hence show that this may be written in the form,

$$R_{\text{rad}}^2 = N_2 A_{21} [\bar{n} + 1]$$

where $\bar{n}$ is the mean number of photons per mode. (Note, you do not need to assume that the radiation is blackbody to prove this.)

(e) Use this result to explain the difference between the number of photons per mode in laser and non-laser sources.

2. Einstein treatment in the absence of line broadening: In this problem we consider the interaction of an atom with radiation which has a bandwidth
5. PROBLEMS

much larger than the linewidths of transitions between the atomic levels, so that we may ignore the complications of line broadening and use the unmodified Einstein coefficients. For the sake of simplicity, we will also assume that the degeneracies of the two levels are equal.

(a) Consider first the situation shown in Fig. 5.1(a) of an atom with two levels only and in which the two levels exchange population only by emission or absorption of radiation. By considering the rate equation for either the upper or lower level, show that under steady-state conditions the population densities are related by,

\[
\frac{N_2}{N_1} = \frac{B_{12} \rho(\omega_{21})}{B_{21} \rho(\omega_{21}) + A_{21}}. \tag{5.3}
\]

(b) What happens to the relative populations in the two levels as the energy density of the radiation is increased to very large values (do not assume that the radiation is blackbody)? Would it be possible to create a population inversion this way?

(c) Now consider the situation shown in 5.1(b) in which the upper and lower levels are pumped at rates \(R_2\) and \(R_1\) and decay with fluorescence lifetimes \(\tau_2\) and \(\tau_1\) respectively. For the remainder of this question we will also ignore spontaneous emission on the transition 2 \(\rightarrow\) 1 (i.e. \(A_{21} \approx 0\)). Show that in the absence of any radiation the steady-state populations in the two levels are given by,

\[
N_2 = R_2 \tau_2 \tag{5.4}
\]

\[
N_1 = R_1 \tau_1, \tag{5.5}
\]

and hence find an expression for the unsaturated population inversion density \(N^*(0) = N_2 - N_1\).

(d) Find the population densities in the two levels when the atom experiences radiation of energy density \(\rho(\omega)\), and hence show that the population inversion density is given by,

\[
N^* = N_2 - N_1 = \frac{N^*(0)}{1 + B_{21} \rho(\omega_{21})[\tau_2 + \tau_1]}. \tag{5.6}
\]
Figure 5.2: Measured full width at half maximum of the homogeneously broadened component of the $D_1$ and $D_2$ lines of Cs as a function of the pressure of helium (data from A. Andalkar and R. B. Warrington *Phys. Rev. A* **65** 032708 (2002)).

(e) Sketch how the population inversion density varies as a function of the energy density of the radiation, and explain the form of your result.

3. Homogeneous broadening:

(a) Explain what is meant by the term **homogeneous broadening**, and briefly describe two examples.

(b) Describe in outline how the natural linewidth of a transition is consistent with the Uncertainty Principle. What is the natural linewidth of a transition between two levels with radiative lifetimes of $\tau_1$ and $\tau_2$?

4. More line broadening: Figure 5.2 shows data from measurements of the homogeneous linewidth of the $D_1$ ($6p^2P_{1/2} \rightarrow 6s^2S_{1/2}$) and $D_2$ ($6p^2P_{3/2} \rightarrow 6s^2S_{1/2}$) transitions in Cs at 894 and 852 nm respectively.

In addition to the homogeneous broadening, inhomogeneous broadening is caused by the thermal motion of the atoms. This so-called Doppler broadening has a Gaussian lineshape with a full-width at half-maximum linewidth given by,

$$\Delta \nu_D = \sqrt{8 \ln 2} \sqrt{\frac{k_B T}{M c^2}} \nu_0,$$

where $T$ is the temperature of the atoms, $M$ their mass, and $\nu_0$ the frequency emitted on the transition by a stationary atom.
5. PROBLEMS

(a) Calculate the Doppler width in MHz of these transitions assuming that the temperature of the Cs vapour is 21°C, and comment on the relative magnitudes of the inhomogeneous and homogeneous linewidths.

(b) The radiative lifetimes of the $D_1 (6p^2P_{1/2}$ and $D_2 (6p^2P_{3/2}$ levels are 34.75 and 30.41 ns respectively. What is the natural linewidth of the $D_1$ and $D_2$ transitions? Is your calculated value consistent with Fig. 5.2?

(c) Use the data presented to deduce the rate of increase in the homogeneous linewidth in units of MHz Torr$^{-1}$ for each of the two transitions. Explain briefly the cause of this increase in homogenous linewidth with pressure.

(d) What is the mean collision time at a He pressure of 100 Torr for He-Cs collisions?

[The molar mass of Cs is 132.9 g.]

5. Steady-state laser oscillation:

(a) The He-Ne laser operates on several $s \rightarrow p$ transitions in neon, including the $5s \rightarrow 3p$ transition at 632.8 nm. Under the operating conditions of the laser, the fluorescence lifetimes of the upper and lower levels are approximately 100 ns and 10 ns respectively for this transition, and the Einstein-A coefficient is $10^7$ s$^{-1}$. Taking the upper and lower laser levels to have equal degeneracies, determine whether or not it is possible, in principle, for continuous-wave laser oscillation to be observed on this transition.

(b) Repeat the calculation for the $3d^{10}4p^2P_{3/2} \rightarrow 3d^{9}4s^2^2D_{5/2}$ transition at 510 nm in the copper-vapour laser, given that the Einstein A coefficient is $2 \times 10^6$ s$^{-1}$ and the fluorescence lifetime of the lower laser level is approximately 10 µs.

6. Optical gain cross-section:

(a) Show that the optical gain cross-section of a homogeneously broadened laser transition may be written as,

$$\sigma_{21}(\omega - \omega_0) = \frac{\pi^2 c^2}{\omega_0^2} A_{21} g_H(\omega - \omega_0), \quad (5.7)$$

where $g_H(\omega - \omega_0)$ is the lineshape function of the transition.

(b) Show that for the special case of a purely lifetime broadened transition from an upper level 2 which decays only radiatively to a long-lived lower level 1 the peak optical gain cross-section is given by:

$$\sigma_{21}(0) = \frac{\lambda_0^2}{2\pi}, \quad (5.8)$$

where $\lambda_0$ is the vacuum wavelength of the transition.
(c) A laser operates on a transition from an excited electronic energy level of a diatomic molecule in which the two constituent atoms form a molecular bond with each other. This level has a lifetime of 10 ns against radiative decay which is entirely on the laser transition at 250 nm to the unstable ground electronic level, which has a lifetime of $3 \times 10^{-14}$ s against dissociation into its constituent atoms.

i. Calculate the peak optical gain cross-section of the laser transition, assuming that it is purely lifetime broadened.

ii. What upper level population density would be needed to provide a small signal gain of $0.1 \text{cm}^{-1}$?

iii. Assuming that 10% of the power input leads to formation of molecules in the upper laser level, calculate the minimum power input per unit volume required to sustain the population of the upper level at the value calculated above. Comment briefly on the implications of this result.

7. Gain saturation:

(a) Use a rate equation analysis to show that the gain coefficient of a homogeneously broadened laser transition is modified by the presence of narrow-band radiation of total intensity $I$ to,

$$\alpha_I(\omega - \omega_0) = \frac{\alpha_0(\omega - \omega_0)}{1 + I/I_s(\omega_L - \omega_0)},$$

where $\omega_L$ is the laser frequency. Give an expression for the saturation intensity $I_s$.

(b) Explain in physical terms why the saturation intensity depends on the detuning of the intense beam from the centre frequency of the transition.

(c) On the same graph plot the gain coefficient as a function of frequency $\omega$:

i. as measured by a weak probe beam in the absence of any other radiation;

ii. as measured by a weak probe beam in the presence of a narrow-band beam of intensity $I_s(\omega_L - \omega_0)$;

iii. as measured by an intense, narrow-band beam of constant intensity $I_s(0)$.

8. A saturated amplifier: A steady-state laser amplifier operates on the homogeneously broadened transition between two levels of equal degeneracy. Population is pumped exclusively into the upper level at a rate of $1.0 \times 10^{18} \text{ s}^{-1} \text{ cm}^{-3}$. The lifetimes of the upper and lower levels are 5 ns and 0.1 ns respectively. A collimated beam of radiation enters the 2 m long amplifier with an initial intensity $I(0)$. The gain cross section of the medium is $4 \times 10^{-12} \text{ cm}^2$ at the 400 nm wavelength of the monochromatic beam. Calculate the intensity of the beam at the exit of the amplifier when:

(a) $I(0) = 0.1 \text{ W cm}^{-2}$
(b) $I(0) = 500 \text{ W cm}^{-2}$

(c) $I(0) = 50 \text{ W cm}^{-2}$

[HINT: In case (c), guess a solution and proceed by iteration]

9. Output power:

(a) Show that the output power of a low-gain, steady-state, homogeneously-broadened laser can be written in the form,

$$P = \frac{1}{2} I_s A_{\text{mode}} T \left( \frac{2\alpha_0 L_g}{T + \delta_{\text{loss}}} - 1 \right),$$

(5.10)

where $T$ is the transmittance of the output coupler.

(b) Find the output coupling $T_{\text{opt}}$ that optimizes the output power, and hence show that the maximum output power can be written as,

$$P_{\text{opt}} = \alpha_0 L_g I_s A_{\text{mode}} \left( 1 - \sqrt{\frac{\delta_{\text{loss}}}{2\alpha_0 L_g}} \right)^2.$$  

(5.11)

(c) Hence show that a loss as small as 10% of the fractional small-signal round-trip gain can decrease the maximum output power to around 50% of that which could be achieved if $\delta_{\text{loss}} = 0$.

10. Three- and four-level lasers:

(a) Explain briefly what is meant by the terms **three-level** and **four-level laser**. Discuss why there is a large difference in the threshold power which is required to achieve laser oscillation in these two classes of laser.

(b) A laser cavity is formed by two mirrors of reflectivity 100% and 95%. Calculate the energy absorbed by the active ions which is necessary to achieve pulsed laser oscillation for rods of ruby and Nd:YAG each 50 mm long, 5 mm diameter, and each with an active ion concentration of $4 \times 10^{19} \text{ cm}^{-1}$. Take the pump bands to be at 20,000 cm$^{-1}$ and 12,000 cm$^{-1}$ for the ruby and Nd:YAG laser respectively.

(c) How will these values compare with the electrical energy which must be supplied to the laser?